

NNRU, a noncommutative analogue of NTRU

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Abstract. NTRU public key cryptosystem is well studied lattice-based Cryptosystem along with Ajtai-Dwork and GGH systems. Underlying NTRU is a hard mathematical problem of finding short vectors in a certain lattice. (Shamir 1997) presented a lattice-based attack by which he could find the original secret key or alternate key. Shamir concluded if one designs a variant of NTRU where the calculations involved during encryption and decryption are non-commutative then the system will be secure against Lattice based attack. This paper presents a new cryptosystem with above property and we have proved that it is completely secure against Lattice based attack. It operates in the non-commutative ring $\mathbf{M} = M_k(\mathbb{Z})[X]/(X^n - I_{k \times k})$, where \mathbf{M} is a matrix ring of $k \times k$ matrices of polynomials in $R = \mathbb{Z}[X]/(X^n - 1)$. Moreover We have got speed improvement by a factor of $O(k^{1.624})$ over NTRU for the same bit of information.

Keywords: public key cryptosystem, NTRU, lattice based cryptosystem

1 Introduction

The first version of NTRU was proposed by (Hoffstein 1996). It has been assessed recently as the fastest public key cryptosystem [1]. Its strong points are short key size, and speed of encryption and decryption. Two assets of crucial importance in embarked application like hand held device and wireless systems . The description of NTRU system is given entirely in terms of quotient ring of Integer polynomials. The most expected attack on this system is Lattice-based attack. The NTRU public key cryptosystem [1] relies for its security on the presumed difficulty of solving the shortest[7,12] and closest vector problem in certain lattices related to the cyclotomic ring $\mathbb{Z}[X]/(X^n - 1)$. Lattices have been studied by cryptographers for quite some time, both in the field of cryptanalysis and as a source of hard problems on which to build encryption schemes[1].

By lattice attack our aim is to find the original key or an alternative key which can be used in place of original key to decrypt ciphertext with some more computational complexity[3]. We construct a lattice whose elements will corresponding to alternative key. If we get a vector as short as original key, we can easily decrypt but even if we find a vector that is two or three times bigger,

we can partially decrypt it by adding the pieces to get the whole. So added security can be achieved by increasing the dimensions of the lattice but it will decrease the speed for encryption and decryption that is the key property of NTRU.

In this paper we present another variant of NTRU, we will call it NNRU. Our focus involves extension to noncommutative groups instead of using group algebra over \mathbb{Z}_n (that is, the ring $\mathbb{Z}_q[X]/(X^n - 1)$).

NNRU operates in the ring of k by k matrices of k^2 different polynomials in $R = \mathbb{Z}[X]/(X^n - 1)$. As matrix multiplication in NNRU is strictly non-abelian. Adversary will have to find out two ring elements. So search space will be square times than that of NTRU. In section 5 we have shown that NNRU is completely secure against lattice attack that was more likely on NTRU and its variants. We can compare an instance of NTRU by putting $n(k^2) = N$. Encryption and decryption in NTRU needs $O(N^2)$ or $O(nk^4)$ operations for a message block on length of N but in NNRU for same bit of information we need $O(nk^{2.376})$ operations if we use coppersmith algorithms for matrix multiplication. that is considerable speed improvement over original NTRU. Inversion of polynomial matrix can be done quickly with less memory-expense by the algorithm suggested in [28]. Moreover polynomial matrix computations can be solved in $\tilde{O}(nk^e)$ by reducing polynomial matrix multiplication to determinant computation and conversely, under the straight line model [27]. Here \tilde{O} denotes some missing $\log(nk)$ factors and e is exponent of matrix multiplication over R .

The paper is organized as follows. Section 2 gives some notation and norm estimation, that help our analysis. In section 3 we briefly sketch NNRU cryptographic system. In section 4 we discuss constraints for parameters. Details of the security analysis of NNRU system is given in sections 5. Section 6 shows performance analysis and comparison with NTRU.

2 Notations

All computations in NNRU are performed in the ring $\mathbf{M} = M_k(\mathbb{Z})[X]/(X^n - I_{k \times k})$, where \mathbf{M} is a matrix ring of $k \times k$ matrices of elements in the ring $R = \mathbb{Z}[X]/(X^n - 1)$. An element $a_0 + a_1x + \dots + a_{n-1}x^{n-1}$ of R can be represented as n -tuple of integers $[a_0, a_1, \dots, a_{n-1}]$. Addition in R is performed componentwise, and multiplication is a circular convolution.

2.1 Norm Estimation

We define width of an element $M \in \mathbf{M}$ to be

$$\|M\|_\infty = \text{Max}(\text{coeff.in polys. } m \in M) - \text{Min}(\text{coeff.in polys. } m \in M)$$

The width of matrices $M \in \mathbf{M}$ is difference between maximum and minimum coefficient in any of k^2 polynomials of it. We say a matrix $M \in \mathbf{M}$ is short if

$$\|M\|_\infty \leq p.$$

The width of the product of two matrices is also be short as it is very less than q , though it may be slightly more than p . We define width of the polynomial $r \in R$ to be

$$\|r\|_\infty = \text{Max}(\text{coeff. in } r) - \text{Max}(\text{coeff. in } -r)$$

Similarly the polynomial r is said to be short if

$$\|r\|_\infty \leq p.$$

Basically width of M or r is a sort of L^∞ norm on \mathbf{M} or R respectively. In this paper we are essentially using all calculation on the L^2 norm to produce an estimate of its L^∞ norm. For precisely evaluating the properties we need to estimate L^∞ but L^2 norm is comparatively easy to estimate. We are giving a proposition between L^∞ and L^2 norm by which we can do all calculations on L^2 norm and estimate on L^∞ norm. It is based on experiments and suggestions due to Don Coppersmith[1]

Let $\|r\|$ be the L^2 norm for a random polynomials r . Then following proposition is true for random polynomials $r_1, r_2 \in R$ with small coefficients .

$$\begin{aligned} \|r_1 * r_2\| &\approx \|r_1\| \cdot \|r_2\| \\ \text{and } \|r_1 * r_2\|_\infty &\approx \gamma \|r_1\| \cdot \|r_2\| \text{ where, } \gamma < 0.15 \text{ for } n < 1000. \end{aligned} \quad (1)$$

Now we define a centered L^2 norm on \mathbf{M} . We denote it by the notation $\|M\|$.

$$\|M\| = \sqrt{\sum_{(\text{polys. } m \in M)} \sum (\text{Coeff. in } m - \mu)^2}$$

where $\mu = \frac{1}{nk^2} \left(\sum_{(\text{polys. } m \in M)} \sum (\text{Coeff. in } m) \right)$ is the average of all coefficient in all the polynomial in matrices M . Its value will be close or equal to zero. Equivalently $\|M\|/\sqrt{nk^2}$ is standard deviation of the coefficients of the polynomials in $M \in \mathbf{M}$. In this paper we do analysis on L^2 centered norm of M and can deduce results on L^∞ norm by using result (1).

The proposition (1) can be extended to the centered L^2 norm on \mathbf{M} . Consider any $\kappa > 0$ there are constants $\gamma_1, \gamma_2 > 0$ and two matrices $M_1, M_2 \in \mathbf{M}$ We therefore express

$$\begin{aligned} \|M_1 * M_2\| &\approx \|M_1\| \cdot \|M_2\| \\ \text{and } \gamma_1 \|M_1\| \cdot \|M_2\| &\leq \|M_1 * M_2\|_\infty \leq \gamma_2 \|M_1\| \cdot \|M_2\| \end{aligned} \quad (2)$$

On the basis of experimental evidence and due to Don Coppersmith[1], The proposition holds good with probability greater than $1 - \kappa$ for small κ . It can be shown experimentally that even for larger value of nk^2 , the value of γ_1/γ_2 is somewhat between zero and one (moderately larger than zero).

2.2 Sample Spaces

NNRU cryptosystem depends on four positive integer parameters (n, k, p, q) with p and q relatively prime and four sets of matrices $(L_f, L_c, L_\phi, L_m) \subset \mathbf{M}$. Note that q will always be considered much larger than p . In this paper, for ease of explanation, we stick to $p = 2$ or 3 , and q ranges between 2^8 to 2^{11} . When we do Matrix multiplication modulo p (or q), we mean to reduce the coefficients of the polynomial in matrices modulo p (or q).

The set of matrices (L_f, L_c, L_ϕ, L_m) consists of all matrices of polynomials in the ring $R = \mathbb{Z}[X]/(X^n - 1)$. The set of matrices (L_f, L_c, L_ϕ) contains polynomials from the set of polynomials $L(d_1, d_2)$

$$L(d_1, d_2) \stackrel{\text{def}}{=} \{u \in R \mid u \text{ has } d_1 \text{ coeff. equal } 1, d_2 \text{ coeff. equal } -1, \text{ and rest } 0\}.$$

where, $d_1 = d_2 < n/2$ or $d_1 = d_2 \approx n/p$

The space of message L_m consists of all matrices of polynomials with coefficients modulo p . We therefore express

$$L_m \stackrel{\text{def}}{=} \{M \in \mathbf{M} \mid \text{polynomial in } M \text{ has coeff. lying between } -\frac{p-1}{2} \text{ and } \frac{p-1}{2}\}.$$

Here we explain individually the meaning and compositions of the all four sets of matrices $(L_f, L_c, L_\phi, L_m) \subset \mathbf{M}$:

1. L_f with elements f and g , and L_ϕ with elements ϕ consist of small matrices of polynomials f and g , are used to compose private key while ϕ will be used as blinding value for each encryption. L_f must satisfy the requirement to have inverse modulo p and modulo q .
2. element w and c belongs to matrix set L_w and L_c respectively. L_c should satisfy the requirement that to have inverse modulo p . w and c are used to construct public key.
3. the set of message L_m consist of matrices of polynomials with coefficients modulo p . We therefore express

$$L_m = \left\{ M \in \mathbf{M} \mid (\text{Polys.}) \text{ (in) } M \in \left(-\frac{p-1}{2} \dots \dots \frac{p-1}{2} \right)^n \subseteq R \right\}$$

3 The NNRU System

3.1 Key Creation

To create a NNRU public/private key pair Bob randomly chooses $f, g \in L_f$ and $w \in L_w$ and $c \in L_c$. Matrices f must satisfy additional requirement to have inverse modulo p and q . Matrices g and c should have inverse modulo p . We denote these inverses by notation F_p, F_q, G_p, C_p respectively.

$$f F_q \equiv I(\text{mod } q) \quad \text{and} \quad g G_p \equiv I(\text{mod } p)$$

$$G_q g \equiv I(\text{mod } q) \quad \text{and} \quad C_p c \equiv I(\text{mod } p)$$

Bob next computes the matrices

$$h \equiv wG_q \pmod{q} \tag{3}$$

$$H \equiv F_q c \pmod{q} \tag{4}$$

Bob publish the pair of matrices $(h, H) \in \mathbf{M}$ as his public key, retaining (f, g, c) as his private key. Polynomial C_p and G_p is simply stored for later use.

3.2 Encryption

Suppose Alice(the encryptor)wants to send a message to Bob (the decryptor). Alice selects a message m from the set of plaintext L_m . Next, Alice randomly choose a matrices $\phi \in L_\phi$ and use, Bob's public key (h, H) to compute (the ciphertext e)

$$e \equiv p\phi h + Hm \pmod{q}$$

Alice then transmit e to Bob. A different random choices of blinding value ϕ is made for each plaintext m .

3.3 Decryption

To decrypt the cipher text, Bob first compute

$$A \equiv feg \pmod{q}$$

$$A \equiv f(p\phi h + Hm)g \pmod{q}$$

$$A \equiv fp\phi hg + fHmg \pmod{q}$$

$$A \equiv pf\phi wG_q g + fF_q cmg \pmod{q}$$

$$A \equiv pf\phi w + cmg \pmod{q}$$

Where he choose the coefficients of the polynomials of the matrices A to lie in interval of $-q/2$ to $q/2$. Why decryption works? Matrices ϕ, g, f, m, c and w have polynomials with small coefficients and p is much smaller than q . It is highly probable for the appropriate parameter choice of the members, matrices $pf\phi w + cmg$, before reducing mod q , has polynomials with coefficients of absolute value less than $q/2$. Bob next computes the matrices B

$$B \equiv A(\text{mod } p)$$

$$B \equiv cmg(\text{mod } p)$$

He reduces each coefficient of the element of A to modulo p . Finally Bob uses his other private keys C_p and G_p to recover the original message.

$$C \equiv C_p cmg G_p(\text{mod } p)$$

$$C \equiv m(\text{mod } p)$$

The matrix C will be the original message m as

$$\text{polynomial in } m \in \left(-\frac{p-1}{2}, \dots, \frac{p-1}{2}\right)^n \subseteq R$$

4 Parameter Constraint

Our selection is based on the following three requirements

1. $f\phi w$ and cmg should be small in order for decryption to work.
2. Appropriate selection of f , g and c prevent a private key attack.
3. Appropriate selection of ϕ and m prevent plain text attack.

The key point is that decryption will only work if $f\phi w$ and cmg are not too large so we want to keep $|pf\phi w + cmg|_\infty$ should be small. For security reasons, it is important that w , remains secret from attacker. On average $|w| \approx |m|$. this type of selection follows

$$|pf\phi w| \approx |cmg|$$

As already described that we are selecting f , g from L_f , c from L_c and w from L_w , m from L_m which gives $d_1 = d_2 \approx n/p$; that ensure to maximize the number of possible choices for polynomials of these matrices.

5 Cryptanalysis

5.1 Brute Force Attacks

To decrypt the cipher text, attackers need to know the private key f , g and c correctly. Attacker can try all possible $f, g \in L_f$ so that $hg \pmod{q}$ should have polynomials with small entries or by finding all $g \in L_f$ and testing if $fH \pmod{q}$ have polynomial with small entries. Out of these small $fH \pmod{q}$, one will be $c \pmod{q}$. So attacker need to search pair of (f, g) . f and g are determined by $2k^2$ polynomials, each of them having maximum degree $(n-1)$. so the number of possible (f, g) pairs are

$$\text{Key Security} = \left[\frac{n!}{(n-2d_f)!d_f!^2} \right]^{2k^2}$$

Here d_f and d_ϕ are defined by assuming L_f and L_ϕ contains polynomials from the set of polynomials $L(d_f, d_f)$ and $L(d_\phi, d_\phi)$ respectively. By analogy, the same attack can also be done against a given message by testing all possible $\phi \in L_\phi$ and search for the matrices $e - \phi h \pmod{Q}$ which contains polynomials with small entries. So individual message security is defined by

$$\text{Message Security} = \left[\frac{n!}{(n-2d_\phi)!d_\phi!^2} \right]^{2k^2}$$

A meet-in-middle attack was proposed by Andrew Odlyzko [13] for NTRU and developed by Silverman. This attack can also be used against NNRU. The attack need a lot of storage capacity and cut the search time by the square root.

5.2 Multiple Transmission Attack

This attack works if Alice sends a single message m several time using same public key but different blinding values ϕ 's, then the attacker eve can get the maximum bits of the message.

suppose Alice transmit the message

$$e_i \equiv \phi_i h + Hm \pmod{q}$$

for $i = 1, 2, \dots, r$

eve can compute $(e_i - e_1) * h^{-1} \pmod{q}$. therefore recovering $\phi_i - \phi_1 \pmod{q}$. If r is of moderate size (say 5 or 6), eve will recover enough bits of ϕ_1 to apply brute force to the rest of the bits. As polynomial of ϕ have small coefficients so eve will recover exactly $\phi_i - \phi_1$, and in the way eve will recover many of coefficients of polynomial of ϕ_1

due to this attack we suggest not to use multiple transmission with further scrambling of particular (underlying) message. However this attack will work for a single message (that has been multiple transmitted) not for any subsequent message.

5.3 Lattice Attack

The Decryptor computes

$$A = feg \equiv pf\phi w + cmg \pmod{q}$$

parameter are chosen so that both $pf\phi w$ and cmg are small enough to guarantee the entries of non modular expression

$$B = pf\phi w + cmg \pmod{q}$$

lies between $-q/2$ and $q/2$ most of the time. In this case decryptor can switch to compute modulo p from computing modulo q and can calculate message.

$$m \equiv C_p B G_p \pmod{p}$$

we can estimate bounds on the elements of B provided correct decryption. Decryption will work only when B is equal to $pf\phi w + cmg$, not mere congruent to modulo q . Using result(2) we can say the following

$$\| pf\phi w \| \approx p \| f \| \| \phi \| \| w \|$$

$$\| cmg \| \approx \| c \| \| m \| \| g \|$$

Assuming vectors $pf\phi w$ and cmg to be nearly orthogonal, we can write

$$\| B \|^2 \approx p^2 \| f \|^2 \| \phi \|^2 \| w \|^2 + \| c \|^2 \| m \|^2 \| g \|^2 \quad (5)$$

decoding will fail if any coefficient of polynomial of B will more than $q/2$ in absolute value. Make the second assumption that the entries of polynomials in matrices B are normally distributed with mean zero and standard deviation $\sigma \approx \frac{\|B\|}{\sqrt{nk^2}}$. Analogues to shamir's results for NTRU [1], Experiments suggests the fact that the probability of correct decoding is high for small ratio of σ to $q/2$. We can say that reliability of decoding is directly proportional to the ratio of $\sigma \approx \frac{\|B\|}{\sqrt{nk^2}}$ to q

Equation (5) gives an estimate of the value of B in terms of f, w, c and g . Let us consider the case in which attacker can use an alternate matrices f' in place of original f and g' in place of g . Upon calculate from a value of w' from equation (3) and c' from equation (4), an estimate of $\|B'\|$ can be calculated by equation (5). If this $\|B'\|$ is comparable to $\|B\|$, then it is not tough to recover message using f' and g' so consider

$$\|B\|^2 \approx p^2 \|f\|^2 \|\phi\|^2 \|w\|^2 + \|c\|^2 \|m\|^2 \|g\|^2$$

Assume $\|\phi\|$ and $\|m\|$ to be held constant at a typical value, and putting $\lambda = \|m\| / \|p\| \|\phi\|$, putting the value of λ in above equation, we therefore left with

$$\sigma^2 = \frac{\|B'\|^2}{nk^2} \approx \left(\frac{p^2 \|\phi\|^2}{nk^2} \right) (\|f'\|^2 \|w'\|^2 + \lambda^2 \|c'\|^2 \|g'\|^2)$$

We can attack this cryptosystem if we can make a lattice L in which squared norm of an element being

$$\|f\|^2 \|w\|^2 + \|c\|^2 \|g\|^2$$

In other words if we can construct a lattice from public key pair h, H in which vector (fw, cg) lies or if we show vectors fw and cg to be same linear transformation of public key vectors. In following analysis we show that we can't make such lattice that will generated by public key and contain vectors (fw, cg) .

Encrypted message is left multiplied by f and right multiplied by g . fw and cg are produced by following transformation on public keys.

$$T_{f,g}(1) : 1 \mapsto fg$$

We can define $T_{f,g} : M \rightarrow M$ be the linear map

$$h \mapsto fhg \quad \text{or} \quad h \mapsto fw \tag{6}$$

$$H \mapsto fHg \quad \text{or} \quad H \mapsto cg \tag{7}$$

For further analysis Let us consider the definition of a lattice. Let \mathbb{R}^m be the m -dimensional Euclidian space. A lattice in \mathbb{R}^m is the set

$$L(b_1, b_2, b_3, \dots, b_n) = \left\{ \sum_{i=1}^n x_i b_i : x_i \in \mathbb{Z} \right\}$$

of all integer combination of n -linear independent vectors $\{b_1, b_2, b_3, \dots, b_n\}$ in \mathbb{R}^m ($m \geq n$). Here we try to make a Lattice of dimensions $2nk^2 \times 2nk^2$ with basis vectors produced by the cyclic shift of the coefficients of polynomial of the matrices h and H . Attacker can crack the system provided the Lattice contains vector (fw, cg) .

One can conclude by linear transformation shown in equation (6) and (7) that the lattice attack is possible if and only if one can make a lattice with public key vectors (h, H) which contains vector (fw, cg) or if following transformation is linear

$$(h, H) \mapsto (fw, cg) \quad (8)$$

In following analysis we show transformation $h \mapsto fhg$ is not linear. Similarly it follows $H \mapsto fhg$ and $(h, H) \mapsto (fw, cg)$ can not be linear.

Consider the multiplication of the matrices $f.h.g = fw$, where each matrix (f, g, h, fw) having k^2 short polynomials as elements

$$\begin{bmatrix} f_1 & \cdots & f_k \\ \vdots & \ddots & \vdots \\ f_{k(k-1)} & \cdots & f_{k^2} \end{bmatrix} \begin{bmatrix} h_1 & \cdots & h_k \\ \vdots & \vdots & \vdots \\ h_{k(k-1)} & \cdots & h_{k^2} \end{bmatrix} \begin{bmatrix} g_1 & \cdots & g_k \\ \vdots & \vdots & \vdots \\ g_{k(k-1)} & \cdots & g_{k^2} \end{bmatrix} = \begin{bmatrix} fw_{1,1} & \cdots & fw_{1,k} \\ \vdots & \vdots & \vdots \\ fw_{k,1} & \cdots & fw_{k,k} \end{bmatrix}$$

$$\begin{aligned} (fw)_{1,1} &= g_1 f_1 h_1 + g_{k+1} f_1 h_2 + g_{2k+1} f_1 h_3 + \cdots + g_{k(k-1)+1} f_1 h_k + g_1 f_2 h_{k+1} \\ &\quad + \cdots + g_{k(k-1)+1} f_2 h_{2k} + \cdots + g_{k(k-1)+1} f_k h_{k^2} \\ (fw)_{1,2} &= g_2 f_1 h_1 + \cdots + g_{k(k-1)+2} f_k h_{k^2} \\ &\vdots \\ (fw)_{k,k} &= g_k h_1 f_{k(k-1)+1} + g_{2k} h_2 f_{k(k-1)+1} + \cdots + g_{k^2} h_k f_{k(k-1)+1} + \cdots + g_{k^2} h_{k^2} f_{k^2} \end{aligned}$$

So general term can be represented as

$$(fw)_{i,j} = \sum_{l=k(i-1)+1}^{ki} \sum_{s=0}^{k-1} f_l(g_{j+sk}) (h_{(1+s)(l-k(i-1))})$$

or, we can represent $(fw)_{i,j} = \sum f_u g_v h_z = \sum U_z h_z$ where, u, v , and z are according to the relationship shown above,

$$\text{Here } i, j \in [1 \ k^2]; \ u, v \in [1 \ k^2]; \ z \in [1 \ k^4]$$

As all U_z are different so we can not find a row vector $S_i = (s_1, s_2, \dots, s_{k^2})$ that will produce vector fw on multiplying with a Lattice represented by the cyclic shift of the coefficients of polynomial of h . In other words if column vectors $v_1, v_2, \dots, v_{nk^2}$ are the basis of lattice $L(v_1, v_2, \dots, v_{nk^2})$, then we will have to multiply different vector S_i to each column vector v_i to get fw . We therefore conclude

$$fw \neq S_i L(v_1, v_2, \dots, v_{nk^2})$$

Thus we proved that one cannot make a lattice by h and H , which contains the vectors (fw, cg) . So lattice attack will not work for this cryptosystem unlike NTRU[1] and its variants [14].

6 Comparison of Security and Speed of NNRU with Other Variants of NTRU

Many variants of NTRU have been introduced till date. We present NNRU as the only variant of NTRU which operates in non-commutative ring. It is completely secure against Lattice attack. Moreover it gives speed improvement over NTRU. Brief of other variants are as follows.

1. **Variant with non-invertible polynomial** [25]: It operates in ring $\mathbb{Z}[X]/(X^N - 1)$. Size of public key and encryption time is roughly doubled than NTRU. It is likely to be more robust against Lattice attack but not proved.

2. **MaTRU** [14]: It operates in a ring of $k \times k$ matrices of polynomials in $R = \mathbb{Z}[X]/(X^n - 1)$ but decryption is not non-commutative. Speed improvement is achieved by a factor of $O(k)$. It gives no added security against lattice or other attacks in comparison with NTRU.

3. **CTRU** [24]: It is analogue of NTRU, the ring of integers replaced by the ring of polynomials $\mathbb{F}_2[T]$. It has been completely cracked by linear algebra attack.

As [25] is slow and [24] is completely cracked so it is obvious to give more attention to the study of security aspect of MaTRU. Here we present meet-in-middle attack on MaTRU and show that the MaTRU system is not more robust against this attack compare to NTRU. This attack can't be operated on NNRU because calculations involved in decryption are non-commutating. [26] shows meet-in-middle attack on NTRU. We show that similar attack can be applied on MaTRU.

Applying same notations as in [14] let us consider Second block of MaTRU Lattice [14].

$$w \pmod{q} = \begin{bmatrix} \gamma_{0,0} \\ \gamma_{0,1} \\ \vdots \\ \vdots \\ \gamma_{k-1,k-1} \end{bmatrix}^T \begin{bmatrix} \boxed{h} \\ \boxed{h \gg 1} \\ \boxed{h \gg 2} \\ \vdots \\ \boxed{h \gg k^2 - 1} \end{bmatrix}$$

nk^2 coefficients of w can be achieved by multiplying row vector γ to matrix h . Idea is to search for γ in the form $\gamma_1 || \gamma_2$, where γ_1 and γ_2 are each of $nk^2/2$ length with $d/2$ ones and “||” denotes concatenation, and then to match $(\gamma_1 * h)$ against $(-\gamma_2 * h)$, looking for (γ_1, γ_2) so that the corresponding coefficients have approximately the same value. The above relationship can be written as

$$\Rightarrow (\gamma_1 * h)_i = \{0, 1\} - (\gamma_2 * h)_i \pmod{q} \forall_i$$

where, the a_i notation denotes the i^{th} entry in a .

This equation is similar to what we get for NTRU [26].

$$\Rightarrow (f_1 * h)_i = \{0, 1\} - (f_2 * h)_i \pmod{q} \forall_i$$

We can operate the attack same as [26]. Assuming $nk^2 = N$ and d are number of ones in γ . Similar to [26], One can easily find that the expected running time and storage space required for this method (this value is equal to what we get for NTRU) is $\binom{N/2}{d/2} / \sqrt{N}$. Further one can also apply meet-in-middle attack on MaTRU followed by Linear algebra attack. Lattice in [14] can also be represented as modular equation $\gamma(y) * h(y) \equiv w \pmod{q} \pmod{(y^{k^2} - 1)}$. It can also be written as

$$\gamma(y) * h(y) = w + qu$$

where, $u = u_{0,0} + u_{0,1} + \dots + u_{k-1,k-1}y^{k^2-1}$ and, $u_{i,j} \in \mathbb{Z}[X]/(X^n - 1)$. Above system of linear equations consist of $3nk^2 - 1$ variable in $nk^2 - 1$ linear equations. If $nk^2 - 1$ is not fairly large than the system of linear equations can be used to reduce an exhaustive search to a space of size 2^{nk^2-1} . further one can set up a meet in middle search to reduce the running time to $O(2^{(nk^2-1)/2})$.

7 Performance Analysis and Comparison with NTRU

Here we present the theoretical operating specification of NNRU and compare the complexity of different operation with standard NTRU PKCS. NNRU cryptosystem depends on four positive integer parameters (n, k, p, q) with p and q relatively prime and four sets of matrices $(L_f, L_c, L_\phi, L_m) \subset \mathbf{M}$. The properties of NTRU [1] is defined in terms of parameters (N, p, q) . We compare two systems for the same size of plaintext blocks by setting $N = nk^2$.

Characteristics	NTRU	NNRU
Plain text Block	$N \log_2 p$ bits	$nk^2 \log_2 p$ bits
Encrypted Text Block	$N \log_2 q$ bits	$nk^2 \log_2 q$ bits
Encryption Speed	$O(N^2)$ operations	$O(n^2 k^3)$ operations
Message Expansion	$\log_p q$ to 1	$\log_p q$ to 1
Private Key Length	$2N \log_2 p$ bits	$2nk^2 \log_2 p$ bits
Public Key Length	$N \log_2 q$ bits	$2nk^2 \log_2 q$ bits
Lattice Security	$2 \left(\frac{\pi^2 a e^2}{3 N q^2} \right)^{\frac{1}{4}}$	Totally secure against lattice attack

¹Since NNRU perform two-sided multiplication during decryption process, so constant factor will about twice that of standard NTRU

² For message security d_g will be replaced by d for NTRU and d_f to d_ϕ for NNRU Cryptosystem

If we compare the size of public/private key, NNRU needs two public keys each of them is double in length that of NTRU public key while the size of private key is same. NNRU gives significant speed improvement over standard NTRU. We can compare an instance of NTRU by putting $n(k^2) = N$. Encryption and decryption in NTRU needs $O(N^2)$ or $O(nk^4)$ operations for a message block on length of N . In NNRU the same bit of information requires $O(nk^{2.807})$ or $O(nk^{2.376})$ operations if we use Strassen's or coppersmith algorithms for matrix multiplication respectively. We can further reduce the number of operations if we use FFT for polynomial multiplication. In this case it will be as small as $O(k^{2.376} n \log n)$, which is considerable speed improvement over original NTRU. It is faster than RSA which needs $O(N^3)$ operations for encryption and decryption.

8 Conclusion

Our motivation for NNRU results from various suggestions given by Shamir and other researchers in their papers for extensions to non-commutative groups. We studied NTRU over ring $\mathbb{F}_2(T)[X]/(X^n - 1)$ but we found that, the variant [24] is secure against Popov Normal Form attack but completely insecure against linear algebra based attacks. Here we follow group algebra over strictly non-commutative groups. Lattice attack is biggest threat to NTRU. It is expected that new lattice reduction technique will be discovered over time and will be able to reduce number of arithmetic operations involved in it. It is natural to study an analogue of NTRU in the given context and find the possibilities in terms of security against Lattice attack and any improvement in terms of speed. NNRU is completely secure against Lattice attacks with significant speed improvement. Further research can be done in the direction of finding the possibilities of any other type of attack or further improvement and generalization of NNRU Cryptosystem.

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